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Divergencies in the Casimir energy for a medium with realistic ultraviolet behaviour

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Abstract
We consider a dielectric medium with an ultraviolet behaviour as follows from the Drude model. Compared with dilute models, this has the advantage that, for large frequencies, two different media behave in the same way. As a result one expects the Casimir energy to contain fewer divergencies than for the dilute media approximation. We show that the Casimir energy of a spherical dielectric ball contains just one divergent term, a volume one, which can be renormalized by introducing a contact term analogous to the volume energy counterterm needed in bag models.

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1. Introduction

Due to recent progress in experimental techniques, it is now possible to measure Casimir forces between macroscopic testbodies very accurately [1, 2]. Recently, even the relevance of the Casimir effect in microelectromechanical systems has been discussed [3] and the actuation of a micromachined torsional device by the Casimir force has been demonstrated. For these reasons a thorough theoretical understanding of the Casimir effect with realistic media is desirable.

There are essentially two ways to calculate the Casimir energy. One possibility is to sum up retarded van der Waals forces between individual molecules [4, 5]. The second way makes use of quantum field theory in the background of a dielectric medium (see e.g. [6–15]). In this second approach one tries to recover retarded van der Waals forces by calculating the vacuum energy for the electromagnetic field in a dielectric background. The relation between these two approaches is not well established. Only for a dilute ball, up to the second order of a perturbative expansion, have the methods been shown to yield the same answers [16, 17].

Otherwise, the presence of divergencies renders a physical interpretation of the finite parts impossible. Even for a dilute medium, once the full dependence of the singularity on the dielectric constants is considered, the result is not even expressible in terms of known special functions [18]. So the counterterms needed to renormalize the divergent contributions are extremely complicated and the interpretation of the classical model associated with them is unclear.

This problematic situation could have its origin in the fact that the models usually employed in describing dielectric media mostly do not incorporate a realistic frequency-dependent dispersion relation. A direct consequence is an inadequate ultraviolet behaviour which leads to the appearance of divergencies which make it impossible to extract, in a physically reasonable way, a finite value for the Casimir energy.

For those reasons, it seems natural to analyse the divergencies of the Casimir effect in a model with a dispersion relation that shows realistic features, at least in the ultraviolet range. As we will see, once this behaviour is assumed, the pole structure resulting in the ζ-function regularization scheme is in fact very simple. Namely, just one pole exists which is completely analogous to the term absorbed into the bag constant in the bag model. So, introducing only one counterterm, namely a volume energy, the Casimir energy can be rendered finite. Apparently, there is no intrinsic way within the field theory framework to fix the finite part of this counterterm. In principle, as is the case in bag models, it has to be fixed by experiments. However, given the simple structure of the renormalization needed, we feel this is an important step in order to understand the relation between different approaches employed and to extract finite Casimir energies in realistic media.

In the next section we analyse the Casimir energy of a simple model with a frequency-dependent permittivity, derived from the Drude model in the high-frequency approximation. This Casimir energy will be defined through the ζ-function of the model, whose singularities will be worked out in section 3. In section 4 we will establish our conclusions.

2. The model and its Casimir energy

Let us start with a description of the simple model we are going to analyse. We consider a nonmagnetic ($\mu = \mu_0$) dielectric ball of radius a, with a frequency-dependent permittivity given by

$$\epsilon(\omega) = \epsilon_0 \left(1 - \frac{\Omega^2}{\omega^2}\right) \tag{1}$$

immersed in the vacuum of permittivity ϵ_0. This frequency-dependent permittivity is the so-called plasma model, which follows from the Drude model in the high-frequency approximation. The phenomenological parameter Ω is usually referred to as the effective plasma frequency. Given that we are going to concentrate on the pole structure of the Casimir energy for the described setting, this high-frequency model is sufficient for our purposes.

Formally, the Casimir energy of this configuration is defined by summing over the vacuum energies of each mode of the electromagnetic field,

$$E_{\mathrm{Cas}}(a) = \sum_n \frac{1}{2}\hbar\,\omega_n = \frac{\hbar\,c}{2a}\sum_n z_n \tag{2}$$

where the dimensionless quantities $z_n = a\omega_n/c$ are the eigenvalues associated with a radius one ball. As it stands, the sum in (2) is divergent, and a regularization procedure must be employed. To properly define the Casimir energy we will use the zeta function regularization [19–22]. In this scheme, $E_{\mathrm{Cas}}(a)$ is defined as the analytic continuation of the function

$$E_{\mathrm{Cas}}(a) \equiv \frac{\hbar c}{2a}\,\zeta(s)\Big|_{s\to-1} \tag{3}$$

where $\zeta(s)$ is defined through the eigenfrequencies of the electromagnetic field by means of the series

$$\zeta(s) = \sum_n z_n^{-s} \tag{4}$$

which is absolutely and uniformly convergent for $\mathbb{R}(s)$ sufficiently large. As is well known, the ζ-function usually has a pole at $s = -1$, and a suitable renormalization procedure is required. (Notice that $\omega_n^2\left(1 - \frac{\Omega_n^2}{\omega_n^2}\right)$ ($\approx\omega_n^2$ for large ω_n^2) correspond to the eigenvalues of a Laplace-type operator.)

In (4), the eigenfrequencies ω_n must be determined by solving the field equation

$$\triangle \vec{E} + \mu\epsilon \frac{\omega^2}{c^2} \vec{E} = 0 \tag{5}$$

(and similarly for $\vec{B}$), subject to matching conditions appropriate to the interphase between two dielectric media,

$$E_{\theta,\phi}|_{r=a^+} = E_{\theta,\phi}|_{r=a^-} \qquad \left.\frac{1}{\mu_1} B_{\theta,\phi}\right|_{r=a^+} = \left.\frac{1}{\mu_2} B_{\theta,\phi}\right|_{r=a^-}. \tag{6}$$

One can consider the transversal electric (TE) modes, for which the electric field has the form

$$\vec{E}_{l,m} = f_l(r) \vec{L} Y_{l,m}(\theta, \phi) \tag{7}$$

and separately the transversal magnetic modes (TM), with the magnetic field given by

$$\vec{B}_{l,m} = g_l(r) \vec{L} Y_{l,m}(\theta, \phi). \tag{8}$$

Here, as usual,

$$\vec{L} = -\mathrm{i}\vec{r} \times \vec{\nabla}. \tag{9}$$

Imposing the matching conditions in (6) leads to

$$\begin{aligned} & f_l(r = a^+) = f_l(r = a^-) \\ & \partial_r[r f_l(r)]|_{r=a^+} = \partial_r[r f_l(r)]|_{r=a^-} \end{aligned} \tag{10}$$

for the TE modes, and

$$\begin{aligned} & g_l(r = a^+) = g_l(r = a^-) \\ & \left.\frac{1}{\epsilon_0} \partial_r[r g_l(r)]\right|_{r=a^+} = \left.\frac{1}{\epsilon(\omega)} \partial_r[r g_l(r)]\right|_{r=a^-} \end{aligned} \tag{11}$$

for the TM modes.

In order to have a discrete spectrum, one might enclose the system into a large conducting sphere of radius R, sending $R \to \infty$ at a suitable intermediate step. This implies also the following boundary condition at $r = R$ for $f_l(r)$ and $g_l(r)$:

$$f_l(r)|_{r=R} = 0 \qquad \partial_r(r\, g_l(r))|_{r=R} = 0. \tag{12}$$

Alternatively, it is possible to use a formulation in terms of the Jost function of the corresponding scattering problem, where a subtraction of the Minkowski space contribution is performed at the beginning and the infinite volume limit taken implicitly [18].

It is straightforward to see that the eigenfrequencies for TE modes are determined as the zeros of the function (we put $\nu = l + 1/2$)

$$\Delta_\nu^{\mathrm{TE}}(z) = \mathcal{J}_\nu(\bar{z}_1)\{\mathcal{Y}_\nu(\bar{z}_0)\mathcal{J}'_\nu(\bar{z}_2) - \mathcal{J}_\nu(\bar{z}_0)\mathcal{Y}'_\nu(\bar{z}_2)\} - \xi\, \mathcal{J}'_\nu(\bar{z}_1)\{\mathcal{Y}_\nu(\bar{z}_0)\mathcal{J}_\nu(\bar{z}_2) - \mathcal{J}_\nu(\bar{z}_0)\mathcal{Y}_\nu(\bar{z}_2)\} \tag{13}$$

where

$$\mathcal{J}_\nu(w) = w\ j_l(w) = \sqrt{\frac{\pi w}{2}} J_\nu(w)$$
$$\mathcal{Y}_\nu(w) = w\ y_l(w) = \sqrt{\frac{\pi w}{2}} Y_\nu(w) \tag{14}$$

are the Riccati–Bessel functions. Here, we introduced the notation $z = a\,(\omega/c)$, $\bar{z}_1 = z\,n(z)$, $\bar{z}_2 = z\,n_0$, $\bar{z}_0 = z\,R\,n_0/a$, $\xi = n(z)/n_0$, $n_0 = \sqrt{\epsilon_0}$ and $n(z) = \sqrt{\epsilon(\omega)}$.

Similarly, for the TM modes the eigenfrequencies are determined as the zeros of

$$\Delta_\nu^{\mathrm{TM}}(z) = \mathcal{J}_\nu(\bar{z}_1)\{\mathcal{Y}'_\nu(\bar{z}_2)\mathcal{J}'_\nu(\bar{z}_0) - \mathcal{J}'_\nu(\bar{z}_2)\mathcal{Y}'_\nu(\bar{z}_0)\} - \frac{1}{\xi}\,\mathcal{J}'_\nu(\bar{z}_1)\{\mathcal{Y}_\nu(\bar{z}_2)\mathcal{J}'_\nu(\bar{z}_0) - \mathcal{J}_\nu(\bar{z}_2)\mathcal{Y}'_\nu(\bar{z}_0)\}. \tag{15}$$

We will suppose that these functions have only real and simple zeros in the open right half-plane.

In the following we will consider explicitly only the TE modes, since the treatment of the TM modes is completely analogous. Due to the spherical symmetry of the problem, the ζ-function has the general appearance

$$\zeta(s) = \sum_{\nu=3/2}^{\infty} 2\nu \sum_n z_{\nu,n}^{-s} \tag{16}$$

where ν labels the angular momentum and, for a given ν, the index n labels the zeros of $\Delta_\nu^{\mathrm{TE}}(z)$ in (13). We need to construct the analytic continuation of $\zeta(s)$ to $s \approx -1$. As we will see, $\zeta(s)$ has a simple pole at $s = -1$ and $E_{\mathrm{TE}}(a)$, defined as in (3), contains a divergent term which depends on the particular dispersion relation adopted for $\epsilon(\omega)$. The method employed for the analytic continuation has been explained in great detail in, for example, [18, 23–26], and we will simply follow this procedure. However, let us mention that the idea of applying this method in the specific context of dielectric media goes back at least to [27].

For $\mathbb{R}(s)$ large enough, we can represent the ζ-function as an integral in the complex z-plane employing Cauchy's theorem. For the TE modes we have

$$\zeta_\nu(s) := \sum_{n=1}^{\infty} z_{\nu,n}^{-s} = \frac{1}{2\pi\mathrm{i}} \oint_C z^{-s} \frac{\Delta_\nu^{\mathrm{TE}\prime}(z)}{\Delta_\nu^{\mathrm{TE}}(z)}\,\mathrm{d}z \tag{17}$$

where the curve C encloses counterclockwise all the positive zeros of $\Delta_\nu^{\mathrm{TE}}(z)$. This kind of representation holds in the presence of arbitrary frequency dispersion, as has already been noted in [14] (see also [28]). The curve C can be deformed into a straight vertical line, crossing the horizontal axis at $\mathbb{R}(z) = x$, where x is any value satisfying $0 < x < z_{\nu,1}$, $z_{\nu,1}$ being the first positive zero of $\Delta_\nu^{\mathrm{TE}}(z)$. Obviously the integral in (17) does not depend on the particular value of x in this range.

Indeed, expressing the integrand in terms of the modified Bessel functions and taking into account their asymptotic behaviour for large arguments, it is easily seen that the integral

$$\zeta_\nu(s) = \frac{-1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty+x}^{\mathrm{i}\infty+x} z^{-s} \frac{\Delta_\nu^{\mathrm{TE}\prime}(z)}{\Delta_\nu^{\mathrm{TE}}(z)}\,\mathrm{d}z \tag{18}$$

converges absolutely and uniformly to an analytic function of s in the open half-plane $\mathbb{R}(s) > 1$. Without loss of generality, and for calculational convenience, we will evaluate this function for real arguments, with s taking values in the half-line $s > 1$, from which it can be meromorphically extended to the whole complex s-plane.

Expression (18) can then be written as

$$\zeta_\nu(s) = -\frac{1}{\pi}\,\mathbb{R}\left\{x^{1-s}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}s}\int_0^\infty (y-\mathrm{i})^{-s}\,\frac{\Delta_\nu^{\mathrm{TE}\prime}(\mathrm{i}x(y-\mathrm{i}))}{\Delta_\nu^{\mathrm{TE}}(\mathrm{i}x(y-\mathrm{i}))}\,\mathrm{d}y\right\} \tag{19}$$

where the prime denotes the derivative with respect to the argument, and we have made use of the properties of the Bessel functions of complex argument.

Changing the integration variable to $t \equiv z(y-\mathrm{i})$, with $z = x/\nu > 0$, we finally get

$$\zeta_\nu(s) = -\frac{1}{\pi}\,\mathbb{R}\left\{\nu^{-s}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(s+1)}\int_{-\mathrm{i}z}^{\infty-\mathrm{i}z} t^{-s}\,\frac{\mathrm{d}(\ln\Delta_\nu^{\mathrm{TE}}(\mathrm{i}\nu t))}{\mathrm{d}t}\,\mathrm{d}t\right\}. \tag{20}$$

Notice that the right-hand side of (20) does not depend on z for $z(>0)$ small enough.

In order to construct the analytic extension of the expression in (3) to $s \approx -1$, we subtract and add the first few terms of the asymptotic expansion of the integrand in (20) (obtained from the uniform asymptotic Debye expansion for the modified Bessel functions appearing in $\Delta_\nu^{\mathrm{TE}}(\mathrm{i}\nu t)$). In fact, in order to isolate the singularities of the Casimir energy it is sufficient to retain only terms up to the order ν^{-3},

$$\frac{\mathrm{d}(\ln\Delta_\nu^{\mathrm{TE}}(\mathrm{i}\nu t))}{\mathrm{d}t} = D_\nu^{\mathrm{TE}}(t) + \mathcal{O}(\nu^{-4}) \tag{21}$$

with

$$D_\nu^{\mathrm{TE}}(t) = \nu D_{\mathrm{TE}}^{(1)}(t) + D_{\mathrm{TE}}^{(0)}(t) + \nu^{-1}D_{\mathrm{TE}}^{(-1)}(t) + \nu^{-2}D_{\mathrm{TE}}^{(-2)}(t) + \nu^{-3}D_{\mathrm{TE}}^{(-3)}(t) \tag{22}$$

where $D_{\mathrm{TE}}^{(k)}(t)$, $k = 1, \ldots, -3$, are algebraic functions of t given in the appendix. Notice that we have also discarded terms which are exponentially vanishing for $R \to \infty$.

Next we must consider the remaining angular momentum sum

$$\begin{aligned}\sum_{\nu=3/2}^{\infty} \nu\mathbb{R}&\left\{\frac{-\nu^{-s}}{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(s+1)}\int_{-\mathrm{i}z}^{\infty-\mathrm{i}z} t^{-s}\,\frac{\mathrm{d}(\ln\Delta_\nu^{\mathrm{TE}}(\mathrm{i}\nu t))}{\mathrm{d}t}\,\mathrm{d}t\right\}\\
&= -\sum_{\nu=3/2}^{\infty}\nu\,\mathbb{R}\left\{\frac{\nu^{-s}}{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(s+1)}\int_{-\mathrm{i}z}^{\infty-\mathrm{i}z} t^{-s}\,D_\nu^{\mathrm{TE}}(t)\,\mathrm{d}t\right\}\\
&\quad -\sum_{\nu=3/2}^{\infty}\nu\,\mathbb{R}\left\{\frac{\nu^{-s}}{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(s+1)}\int_{-\mathrm{i}z}^{\infty-\mathrm{i}z} t^{-s}\left\{\frac{\mathrm{d}(\ln\Delta_\nu^{\mathrm{TE}}(\mathrm{i}\nu t))}{\mathrm{d}t} - D_\nu^{\mathrm{TE}}(t)\right\}\mathrm{d}t\right\}.\end{aligned} \tag{23}$$

The second term in the right-hand side of (23) converges for $s > -2$ by construction. Therefore, we can put $s = -1$ inside the sum and the integral, and numerically evaluate this contribution when necessary. In the next section we will identify the singularities contained in the first term.

3. Ultraviolet singular behaviour of the Casimir energy

In order to investigate the ultraviolet divergencies in E_{Cas} it is sufficient to consider the first term in the right-hand side of (23). It should be stressed that the real part in the argument of the series in the first term in the right-hand side of (23) is in fact independent of z: Taking into account the analyticity of the integrand, one can investigate the z-dependence by studying the integral

$$\int_{-\mathrm{i}z}^{1} t^{-s}\,D_\nu^{\mathrm{TE}}(t)\,\mathrm{d}t \tag{24}$$

which can be exactly solved in terms of hypergeometric functions. It is straightforward to verify that the z-dependent part is imaginary for all $s > 1$, and therefore is dropped out when taking the real part in (23). This feature will be useful in what follows.

For $s > 1$ we can study each term in $D_\nu^{\mathrm{TE}}(t)$ separately, and evaluate the expressions

$$\sum_{\nu=3/2}^{\infty} \nu^{-(s-k-1)} \mathbb{R} \left\{ \frac{-\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(s+1)}}{\pi} \int_{-\mathrm{i}z}^{\infty-\mathrm{i}z} t^{-s}\, D_{\mathrm{TE}}^{(k)}(t)\,\mathrm{d}t \right\} \tag{25}$$

with the index $k = 1, \ldots, -3$ corresponding to the order in the Debye expansion.

Since the real part of the expression between brackets in (25) is independent of ν (it is independent of $z = x/\nu$—see above), the sum over ν can be performed by means of the Hurwitz zeta function,

$$\sum_{\nu=3/2}^{\infty} \nu^{-(s-k-1)} = \zeta_{\mathrm{H}}(s-k-1, 1/2) - 2^{s-k-1}. \tag{26}$$

Only for $k = -3$ does one get a singularity at $s = -1$, which is evaluated by

$$\zeta_{\mathrm{H}}(s+2, 1/2)_{|s\simeq-1} = \frac{1}{s+1} + (\gamma + \ln(4)) + \mathcal{O}(s+1). \tag{27}$$

For the other values of k a regular extension to $s = -1$ is obtained.

Therefore, it is only for $k = -3$ that we need to calculate both finite and singular parts of the analytic extension of the integral in (25) around $s = -1$, while for the other values of k only the singular terms are needed.

All these terms can be worked out exactly. For the divergent part of the contribution of the TE modes to the Casimir energy (in the limit $R \to \infty$) we obtain

$$E_{\mathrm{TE}}(a) = \frac{\hbar c}{a} \left\{ -\frac{{n_0}^3\, a^4 \Omega^4}{24\,\pi\, c^4\,(1+s)} - \frac{n_0 a^2 \Omega^2}{4\pi c^2 (1+s)} + \mathcal{O}(s+1)^0 \right\}. \tag{28}$$

Notice that while the first term in the right-hand side corresponds to a volume contribution and the second one is a curvature contribution, no surface contribution appears. This could be expected since, as is clear from the expression of the permittivity in (1), and for dimensional reasons, the systematic expansion parameter is $(\Omega a)^2$ and odd powers do not appear.

For the TM modes one gets similarly

$$E_{\mathrm{TM}}(a) = \frac{\hbar c}{a} \left\{ -\frac{{n_0}^3\, a^4\, \Omega^4}{24\,\pi\, c^4\,(1+s)} + \frac{n_0 a^2 \Omega^2}{4\pi c^2 (1+s)} + \mathcal{O}(s+1)^0 \right\}. \tag{29}$$

Finally, adding up the contributions coming from the TE and TM modes, we get for the singular piece of the Casimir energy of the dielectric ball just a volume contribution,

$$E(a) = \frac{\hbar c}{a} \left\{ -\frac{{n_0}^3\, a^4 \Omega^4}{12\pi c^4 (1+s)} + \mathcal{O}(s+1)^0 \right\} \tag{30}$$

since the divergent curvature terms have cancelled out between TE and TM modes. Rewriting this result involving explicitly the volume of the ball, $V = 4\pi a^3/3$, we find

$$E(a) = -\frac{\hbar n_0^3 \Omega^4}{16\pi^2 c^3 (1+s)}\, V + \mathcal{O}(s+1)^0. \tag{31}$$

4. Conclusions

In the previous section we found that the pole structure of the Casimir energy in the framework of the ζ-function regularization scheme is very simple, once a realistic behaviour of the dielectric medium at high frequencies is assumed. The pure volume divergence might be seen to represent a contribution to the mass density of the material [16] and it is to be absorbed into a phenomenological counterterm, much in the way as is done in bag models. Neither a surface tension nor a curvature counterterm is needed in this model, since surface divergencies are absent and the curvature ones nicely cancel out between TE and TM modes.

Given this simple pole structure it makes sense to analyse further finite parts of the Casimir energy in a quantum field theory context for realistic media. As a first step one might assume the Drude model for the medium and analyse how the Casimir energy depends on, for example, the plasma and the relaxation frequency. More ambitiously, the dielectric constants might be obtained from tabulated refractive indices using the Kramer–Kronig relation [29]. This is under consideration and hopefully will allow a detailed analysis of the Casimir energy for a spherically symmetric situation with realistic media.

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Appendix. Debye expansion for the TE and TM modes

The functions $D_{\rm TE}^{(k)}(t), k = 1, \ldots, -3$, coefficients of the expansion $D_\nu^{\rm TE}(t)$ (see equations (21) and (22)), which are obtained from the uniform asymptotic Debye expansion for the modified Bessel functions appearing in $\Delta_\nu^{\rm TE}({\rm i}\nu t)$, are given by

$$D_{\rm TE}^{(1)}(t) = \frac{\sqrt{1 + {n_0}^2\, R^2\, t^2/a^2}}{t} \tag{32}$$

$$D_{\rm TE}^{(0)}(t) = \left(2\,t + \frac{2\,{n_0}^2\, R^2\, t^3}{a^2}\right)^{-1} \tag{33}$$

$$D_{\rm TE}^{(-1)}(t) = \frac{{n_0}^2\, R^2\, t\,(1 - {n_0}^2\, R^2\, t^2/4\,a^2)}{2\,a^2\,(1 + {n_0}^2\, R^2\, t^2/a^2)^{5/2}} - \frac{({n_0}^2 + 2/t^2)\, Z^2}{2\,t\,\sqrt{1 + {n_0}^2\, t^2}} \tag{34}$$

$$\begin{aligned} D_{\rm TE}^{(-2)}(t) = \frac{a^6\,{n_0}^2\, R^2\, t}{2\,(a^2 + {n_0}^2\, R^2\, t^2)^4} \Bigg(&+ \frac{5\,{n_0}^2\, t^2}{2} - \frac{{n_0}^4\, R^4\, t^4}{4\,a^4} - \frac{6\,{n_0}^2\, R^2\, Z^2}{a^2} \\ &- \frac{a^2\, Z^2}{{n_0}^2\, R^2\, t^4} - \frac{4\,Z^2}{t^2} - \frac{4\,{n_0}^4\, R^4\, t^2\, Z^2}{a^4} - \frac{{n_0}^6\, R^6\, t^4\, Z^2}{a^6} - 1\Bigg) \end{aligned} \tag{35}$$

and

$$D_{\rm TE}^{(-3)}(t) = \frac{{n_0}^2\,R^2\,t\,(64\,a^6 - 560\,a^4\,{n_0}^2\,R^2\,t^2 + 456\,a^2\,{n_0}^4\,R^4\,t^4 - 25\,{n_0}^6\,R^6\,t^6)}{128\,(1+{n_0}^2\,R^2\,t^2/a^2)^{11/2}}$$

$$+\frac{Z^2\,(16\,Z^2 + 56\,{n_0}^2\,t^2\,Z^2 + 6\,{n_0}^8\,t^8\,Z^2 + 3\,{n_0}^6\,t^6\,(t^2+12\,Z^2) - 2\,{n_0}^4\,(t^6 - 35\,t^4\,Z^2))}{16\,t^5\,(1+{n_0}^2\,t^2)^{7/2}}. \quad (36)$$

Similarly, for the TM modes the functions $D_{\rm TM}^{(k)}(t)$, $k = 1, \ldots, -3$, appearing in the expansion $D_\nu^{\rm TM}(t)$, are given by

$$D_{\rm TM}^{(1)}(t) = \frac{\sqrt{1+{n_0}^2\,R^2\,t^2/a^2}}{t} \quad (37)$$

$$D_{\rm TM}^{(0)}(t) = \frac{-1}{2\,(t+2\,{n_0}^2\,R^2\,t^3/a^2)} \quad (38)$$

$$D_{\rm TM}^{(-1)}(t) = \frac{-({n_0}^2\,R^2\,t\,(8\,a^2+{n_0}^2\,R^2\,t^2))}{8\,(a^2+{n_0}^2\,R^2\,t^2)^2\,\sqrt{1+{n_0}^2\,R^2\,t^2/a^2}} - \frac{(2+{n_0}^2\,t^2)\,Z^2}{2\,t^3\,\sqrt{1+{n_0}^2\,t^2}} \quad (39)$$

$$D_{\rm TM}^{(-2)}(t) = \frac{{n_0}^2\,R^2\,t\,(10-10\,{n_0}^2\,R^2\,t^2/a^2+{n_0}^4\,R^4\,t^4/a^4)}{8\,a^2\,(1+{n_0}^2\,R^2\,t^2/a^2)^4} + \frac{Z^2}{2\,t^3} \quad (40)$$

and

$$D_{\rm TM}^{(-3)}(t) = \frac{-({n_0}^2\,R^2\,t\,(176\,a^6 - 784\,a^4\,{n_0}^2\,R^2\,t^2 + 480\,a^2\,{n_0}^4\,R^4\,t^4 - 23\,{n_0}^6\,R^6\,t^6))}{128\,a^8\,(1+{n_0}^2\,R^2\,t^2/a^2)^{11/2}}$$

$$+\frac{t^2\,(2+{n_0}^2\,t^2)\,(4+12\,{n_0}^2\,t^2+3\,{n_0}^4\,t^4)\,Z^2 + 2\,(1+{n_0}^2\,t^2)^2\,(8+12\,{n_0}^2\,t^2+3\,{n_0}^4\,t^4)\,Z^4}{16\,t^5\,(1+{n_0}^2\,t^2)^{7/2}}. \quad (41)$$

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